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On optimal polynomial meshes

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Abstract

Let P_n^d be the space of real algebraic polynomials of d variables and degree at most n, $K \subset \mathbb{R}^d$ a compact set, $\|p\|_K := \sup_{\mathbf{x}\in K} |p(\mathbf{x})|$ the usual supremum norm on K, and $card(Y)$ the cardinality of a finite set Y. A family of sets $\mathbf{Y} = \{Y_n \subset K, n \in \mathbb{N}\}$ is called an ***admissible mesh*** in K if there exists a constant $c_1 > 0$ depending only on K such that

$$\|p\|_K \le c_1 \|p\|_{Y_n}, \quad p \in P_n^d, n \in \mathbb{N},$$

where the cardinality of Y_n grows at most polynomially. If $card(Y_n) \le c_2 n^d$, $n \in \mathbb{N}$ with some $c_2 > 0$ depending only on K then we say that the admissible mesh is **optimal**. This notion of admissible meshes is related to **norming sets** which are widely used in the literature. In this paper we present some general families of sets possessing admissible meshes which are optimal or near optimal in the sense that the cardinality of sets Y_n does not grow too fast. In particular, it will be shown that graph domains bounded by polynomial graphs, convex polytopes and star like sets with C^2 boundary possess optimal admissible meshes. In addition, graph domains with piecewise analytic boundary and any convex sets in $\mathbb{R}^2$ possess *almost* optimal admissible meshes in the sense that the cardinality of admissible meshes is larger than optimal only by a $\log n$ factor.

Keywords: Multivariate polynomials; Norming sets; Admissible optimal meshes; Graph domains; Convex and star like sets

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1. Introduction

Consider the space P_n^d of real algebraic polynomials of d variables and degree at most n. Let $K \subset \mathbb{R}^d$ be any compact set. Denote by $\|p\|_K := \sup_{\mathbf{x}\in K} |p(\mathbf{x})|$ the usual supremum norm on K. Moreover, $\operatorname{card}(Y)$ stands for the cardinality of a finite set Y.

Definition 1. A family of sets $\mathbf{Y} = \{Y_n \subset K,\ n \in \mathbb{N}\}$ is called an *admissible mesh* in K if there exist constants c_1, c_2 depending only on K such that

$$\|p\|_K \le c_1 \|p\|_{Y_n}, \quad p \in P_n^d,\ n \in \mathbb{N}, \tag{1}$$

where the cardinality of Y_n grows at most polynomially, i.e. $\operatorname{card}(Y_n) \le c_2 n^m,\ n \in \mathbb{N}$ with some fixed $m \in \mathbb{N}$ depending only on K.

This notion of admissible meshes is related to *norming sets* widely used in the literature. (See for instance Jetter et al. [8,9], where norming sets were used for the study of scattered data interpolation and cubature formulas on spheres.) In the present form the notion of admissible meshes appears in a recent paper by Calvi and Levenberg [5], where their application for least squares approximation is discussed. (Some related results for univariate polynomials with application to discretization of best Chebyshev approximation problem can be found in [6, p. 91–95].) Recently, in [2] it was shown that discrete extremal sets of Fekete and Leja type can be extracted from admissible meshes, while in [3] the authors considered low cardinality admissible meshes on some standard compact sets in $\mathbb{R}^2$, for instance disks, triangles. (A survey of some recent results on admissible meshes can be found in [4].)

Since $\dim P_n^d \sim n^d$ we clearly must have $m \ge d$ in the above definition, provided that no polynomial vanishes on K. Of course, in optimal case we aim for a mesh with asymptotically minimal number of points in it, that is we would like to have $m = d$. This leads to the following definition.

Definition 2. We shall say that an admissible mesh $\mathbf{Y} = \{Y_n \subset K,\ n \in \mathbb{N}\}$ in $K \subset \mathbb{R}^d$ is *optimal* if $\operatorname{card}(Y_n) \le cn^d,\ n \in \mathbb{N}$ with some $c > 0$ depending only on K.

The basic question in this respect consists in describing those sets $K \subset \mathbb{R}^d$ which possess optimal admissible meshes. Finding exact geometric properties characterizing sets with optimal admissible meshes appears to be a rather complex problem.

Our goal in this paper is to present sufficiently wide families of sets possessing admissible meshes which are optimal or near optimal in the sense that the cardinality of sets Y_n in the mesh $\mathbf{Y}$ does not grow too fast.

We shall give a systematic study of this question by considering two different categories of domains:

(A) *sets with certain analytic properties, i.e., graph domains bounded by graphs of polynomial, differentiable or analytic functions*;

(B) *sets satisfying certain geometric properties, that is convex bodies, polytopes or star like domains*.

In particular, it will be shown that graph domains bounded by polynomial graphs, convex polytopes and star like sets with C^2 boundary possess optimal admissible meshes. In addition, we shall verify that graph domains in $\mathbb{R}^d$ with piecewise analytic boundary and convex sets in $\mathbb{R}^2$ possess *almost* optimal admissible meshes in the sense that the cardinality of admissible meshes differs from the optimal only by a $\log n$ factor. (For convex bodies in $\mathbb{R}^d, d > 2$ a somewhat weaker result will be proved.) Clearly, taking affine transformations or finite unions of the above-mentioned sets preserves the existence of admissible meshes with same cardinalities. The methods used in the paper are constructive, thus they lead to explicit algorithms for finding good admissible meshes. These methods will rely heavily on some new Bernstein–Markov type inequalities for multivariate polynomials proved in the past decade in [10,11,13]. On the other hand, it will be also shown in the last section of the paper that even though Bernstein–Markov type inequalities provide useful tools for the construction of admissible meshes there exist domains which do not have good Bernstein–Markov properties but nevertheless possess admissible meshes of low cardinality.

In the first section of the paper we present new results on admissible meshes in graph domains, while the second part contains results concerning admissible meshes in convex and star like sets. We conclude the paper by several examples and open problems. Finally, it should be also noted that the results of this paper can be applied for the construction of cubature formulas in general domains in $\mathbb{R}^d$ similarly to [9], where this was accomplished on the unit sphere.

2. Admissible meshes in graph domains

The first result related to the construction of optimal polynomial meshes for univariate polynomials can be found in [6]. It is essentially shown in [6, p. 91, Lemma 3(iii)], that $[-1, 1]$ possesses an optimal admissible mesh $\mathbf{Y} = \{Y_n \subset [-1, 1],\ n \in \mathbb{N}\}$ satisfying (1) with $c_1 = 2$ such that

$$\operatorname{card}(Y_n) \leq [n\pi/2] + 1, \quad n \in \mathbb{N}. \tag{2}$$

(Note that in general, there exist optimal admissible meshes for which (1) holds with any $1 < c_1 \leq 2$ which have cardinality $O(\frac{n}{\sqrt{c_1 - 1}})$.)

Now we shall consider the class of domains bounded by graphs of polynomial, differentiable or analytic functions.

Set $I^k := [0, 1]^k, 1 \leq k \leq d$ and consider arbitrary functions $g_1 \equiv 1, f_1 \equiv 0, 0 \leq f_k(\mathbf{x}) \leq g_k(\mathbf{x}) \leq 1, \mathbf{x} \in I^{k-1}, 2 \leq k \leq d$. Using these functions we can introduce the graph domain

$$\begin{aligned} K_{\mathbf{g}} := \{(x_1, \ldots, x_d) \in \mathbb{R}^d : f_k(x_1, \ldots, x_{k-1}) \leq x_k \\ \leq g_k(x_1, \ldots, x_{k-1}), (x_1, \ldots, x_{k-1}) \in I^{k-1}, 1 \leq k \leq d\}. \end{aligned} \tag{3}$$

When the functions f_k, g_k are algebraic polynomials we shall say that $K_{\mathbf{g}}$ is a polynomial graph domain. Similarly, if f_k, g_k are C^r or analytic functions in an open neighborhood of I^{k-1}, say in $aI^{k-1}, a > 1$, the graph domain is said to be C^r or analytic, respectively.

Theorem 1. *Let $d \geq 2, r \in \mathbb{N}$ and $K_{\mathbf{g}} \subset I^d$ be a C^r graph domain (3) with $0 \leq f_k < g_k \leq 1$ on $I^{k-1},\ 1 \leq k \leq d$. Then $K_{\mathbf{g}}$ possesses an admissible mesh $\mathbf{Y} = \{Y_n \subset K_{\mathbf{g}},\ n \in \mathbb{N}\}$ satisfying (1) such that*

$$\operatorname{card}(Y_n) = O\left(n^{d + \frac{2d(d-1)}{r}}\right), \quad n \in \mathbb{N}. \tag{4}$$

Moreover, if the graph domain is analytic then

$$\operatorname{card}(Y_n) = O(n^d \ln^{d(d-1)} n), \quad n \in \mathbb{N}. \tag{5}$$

It should be noted that by Theorem 1 the cardinality of admissible meshes in C^r-domains approaches the optimal order n^d as r grows, i.e., for smoother graph domains. For analytic graph domains the cardinality becomes almost optimal: it is off from the optimal only by a $\log n$ factor. A similar result for a class of analytic graph domains was recently given in [14].

The next proposition showing that polynomial graph domains possess optimal admissible meshes seems to be of independent interest.

Proposition 1. *Let $d \geq 2, m_1, \ldots, m_d \in \mathbb{N}$ and $K_{\mathbf{g}} \subset \mathbb{R}^d$ be a polynomial graph domain (3) with some $f_k, g_k \in P_{m_k}^{k-1}, 2 \leq k \leq d$. Then $K_{\mathbf{g}}$ possesses an optimal admissible mesh $\mathbf{Y} = \{Y_n \subset K_{\mathbf{g}},\ n \in \mathbb{N}\}$ satisfying (1) with $c_1 = 2^d$ such that* $\operatorname{card}(Y_n) \leq (\pi d)^d (Nn)^d,\ n \in \mathbb{N}$, *where $N := \prod_{k=2}^d (m_k + 1)$.*

Proof of Proposition 1. Let us first verify the proposition in the special case when $K_{\mathbf{g}} = I^d$, i.e., $f_k \equiv 0, g_k \equiv 1, m_k = 0,\ 1 \leq k \leq d$. In this case we can simply take the mesh to be the tensor product of optimal admissible meshes on $[0, 1]$ satisfying (2) and use a standard induction argument to show that for this product mesh having cardinality $([n\pi/2] + 1)^d$ (1) holds with $c_1 = 2^d$.

When $f_k, g_k \in P_{m_k}^{k-1}, 2 \leq k \leq d$ given any $\mathbf{t} = (t_1, \ldots, t_d) \in I^d$ relations $x_1 = t_1,\ x_k = (1 - t_k) f_k(x_1, \ldots, x_{k-1}) + t_k g_k(x_1, \ldots, x_{k-1}), 2 \leq k \leq d$ successively define an $\mathbf{x} = (x_1, \ldots, x_d) \in K_{\mathbf{g}}$. Clearly, $K_{\mathbf{g}}$ is the image of I^d under the mapping $A(\mathbf{t}) = \mathbf{x}$ defined by the above relations. It is easy to show by induction that $x_k(\mathbf{t}) \in P_{dN}^k, 2 \leq k \leq d$ and therefore for any $p \in P_n^d$ we have $p(\mathbf{x}) = p(A(\mathbf{t})) := g(\mathbf{t}) \in P_{dnN}^d$. Thus any admissible mesh on I^d for polynomials in P_{dnN}^d yields and admissible mesh on $K_{\mathbf{g}}$ of the same cardinality for polynomials $p \in P_n^d$. This together with the above observation on cardinality of product meshes on I^d concludes the proof of the proposition. $\square$

The next proposition asserts that perturbing the boundaries of a domain $K_{\mathbf{g}}$ given by (3) by $O\left(\frac{1}{n^2}\right)$ can change uniform norms of polynomials from P_n^d on this domain only by constant multipliers. This is a multivariate extension of the following well-known property of univariate polynomials: for any $p \in P_n^1$ we have

$$\|p\|_{[a,b]} \leq c\|p\|_{[a+1/n^2, b-1/n^2]} \tag{6}$$

with some constant $c > 0$ depending only on $b - a$. (Estimate (6) can be easily deduced, for instance from the Remez inequality, see [1, p. 228].)

Proposition 2. *Let $d \geq 2$ and $K_{\mathbf{g}} \subset I^d$ be a C^1 graph domain (3) with functions $f_k, g_k \in C^1(I^{k-1}), 0 \leq f_k < g_k \leq 1$ on $I^{k-1}, 2 \leq k \leq d, g_1 \equiv 1, f_1 \equiv 0$. Denote by $K_{\mathbf{g},n}$ the graph domain (3) with f_k, g_k being replaced by $f_{k,n} := f_k + 1/n^2, g_{k,n} := g_k - 1/n^2$. Then for any $p \in P_n^d$ we have*

$$\|p\|_{K_{\mathbf{g}}} \leq c\|p\|_{K_{\mathbf{g},n}} \tag{7}$$

with some constant $c > 0$ depending only on

$$\begin{aligned} &\delta_K := \min\{(g_k - f_k)(\mathbf{x}) : \mathbf{x} \in I^{k-1},\ 1 \leq k \leq d\}, \\ &M_K := \max\{(|\partial f_k| + |\partial g_k|)(\mathbf{x}) : \mathbf{x} \in I^{k-1}, 2 \leq k \leq d\}, \end{aligned} \tag{8}$$

where $\partial f_k, \partial g_k$ stands for the gradients of f_k, g_k.

Proof of Proposition 2. We shall verify the proposition by induction on the dimension d.

Let first $d = 2$. Note that for any $(x_1, x_2) \in K_{\mathbf{g},n}$ we have $x_1 \in [1/n^2, 1 - 1/n^2]$ and $f_2(x_1) + 1/n^2 \le x_2 \le g_2(x_1) - 1/n^2$. Hence given $p \in P_n^2$ we can apply estimate (6) for any fixed $x_1 \in [1/n^2, 1 - 1/n^2]$ and univariate polynomials $p(x_1, t) \in P_n^1$ to derive that

$$|p(x_1, x_2)| \le c_1 \|p\|_{K_{\mathbf{g},n}}, \quad x_1 \in [1/n^2, 1 - 1/n^2], \quad f_2(x_1) \le x_2 \le g_2(x_1). \tag{9}$$

(Here and in the remaining part of the proof c_j are positive constants depending only on δ_K, M_K.) Now we need to extend relation (9) to $x_1 \in [0, 1]$. Without loss of generality we may assume that $-f_2(1) = g_2(1) = 1$ and $\|p\|_{K_\mathbf{g}} = p(1, 1)$. (This can be accomplished by linear transformations of the domain involving constants depending only on δ_K.)

Consider the univariate polynomial $q(t) = p(t, M_K(t-1)+1)$. Clearly by (8) for $1 - \frac{1}{M_K} \le t \le 1$ we have

$$f_2(t) \le f_2(1) + M_K(1-t) \le M_K(t-1) + 1 = M_K(t-1) + g_2(1) \le g_2(t).$$

Using (9) we obtain that

$$|q(t)| \le c_1 \|p\|_{K_{\mathbf{g},n}}, \quad 1 - \frac{1}{M_K} \le t \le 1 - 1/n^2.$$

Using the above estimate together with inequality (6) we arrive at

$$\|p\|_{K_\mathbf{g}} = p(1, 1) = q(1) \le c_2 \|p\|_{K_{\mathbf{g},n}}$$

which completes the proof when $d = 2$.

Assume now that the proposition holds for $d - 1$. Fix an arbitrary $x_1 \in [1/n^2, 1 - 1/n^2]$. Whenever $\mathbf{x} = (x_1, \mathbf{z}) \in K_\mathbf{g}$ we have $\mathbf{z} \in K_\mathbf{g}^*$ where $K_\mathbf{g}^*$ is a graph domain (3) in $\mathbb{R}^{d-1}$. Using the induction hypothesis for $K_\mathbf{g}^*$ we obtain that for every $p \in P_n^d$ and $x_1 \in [1/n^2, 1 - 1/n^2]$

$$\|p(x_1, \mathbf{z})\|_{K_\mathbf{g}^*} \le c \|p(x_1, \mathbf{z})\|_{K_{\mathbf{g},n}^*} \le c \|p\|_{K_{\mathbf{g},n}}.$$

Now this last relation can be extended for $x_1 \in [0, 1]$ analogously to the case $d = 2$. $\square$

Proof of Theorem 1. Given $f_k, g_k \in C^r(aI^{k-1})$, $a > 1$ by the multivariate Jackson theorem (see [15]) there exist $p_k, q_k \in P_m^{k-1}$ such that

$$p_k - \frac{c_1}{m^r} < f_k < p_k < q_k < g_k < q_k + \frac{c_1}{m^r}, \quad \mathbf{x} \in aI^{k-1},\ 2 \le k \le d \tag{10}$$

where $c_1 > 0$ depends only on d and $K_\mathbf{g}$. Moreover, denoting by ∂p the gradient vector of p

$$M_K^* := \max\{(|\partial p_k| + |\partial q_k|)(\mathbf{x}) : \mathbf{x} \in I^{k-1}, 1 \le k \le d\} \le c_2 M_K \tag{11}$$

and $\delta_K^* := \min\{(q_k - p_k)(\mathbf{x}) : \mathbf{x} \in I^{k-1}, 1 \le k \le d\} \ge \delta_K/2$ for m large enough. The last statement is an obvious consequence of the continuity of functions involved, but inequality (11) requires some explanation. The fact that the partial derivatives of $p_k, q_k \in P_m^{k-1}$ are uniformly bounded for every $m \ge m_0(K)$ follows from (10), the assumption $f_k, g_k \in C^r(aI^{k-1})$, $a > 1$ and the well-known Stechkin inequality; see Lemma 1 in [12] for details.

Let now $K_\mathbf{q}$ be defined as in (3) with polynomials p_k, q_k replacing functions f_k, g_k, respectively. Similarly, we denote by $K_{\mathbf{q},n}$ the graph domain (3) with polynomials $p_k - 1/n^2$, $q_k + 1/n^2$ replacing p_k, q_k, respectively. Setting $m := [c_1^{1/r} n^{2/r}] + 1$ we have by (10)

$$p_k - \frac{1}{n^2} < f_k < p_k < q_k < g_k < q_k + \frac{1}{n^2}, \quad \mathbf{x} \in I^{k-1},\ 2 \le k \le d. \tag{12}$$

Clearly, (3) and (12) imply that

$$K_{\mathbf{q}} \subset K_{\mathbf{g}} \subset K_{\mathbf{q},n}. \tag{13}$$

Hence by Proposition 2 for any $p \in P_n^d$

$$\|p\|_{K_{\mathbf{g}}} \le \|p\|_{K_{\mathbf{q},n}} \le c_3 \|p\|_{K_{\mathbf{q}}}$$

with some $c_3 > 0$ depending on $K_{\mathbf{g}}$ only.

Now we can apply Proposition 1 to the polynomial graph domain $K_{\mathbf{q}}$ yielding that there exists an admissible mesh $\mathbf{Y} = \{Y_n \subset K_{\mathbf{q}},\ n \in \mathbb{N}\}$ satisfying (1) such that $\operatorname{card}(Y_n) = O((Nn)^d),\ n \in \mathbb{N}$, where $N := (m+1)^{d-1} \le c_5 n^{\frac{2(d-1)}{r}}$. Using (13) and the last estimate combined with (1) we have that $Y_n \subset K_{\mathbf{g}}$ and

$$\|p\|_{K_{\mathbf{g}}} \le c_3 \|p\|_{K_{\mathbf{q}}} \le c_4 \|p\|_{Y_n}, \quad n \in \mathbb{N}$$

i.e., $\mathbf{Y} = \{Y_n \subset K_{\mathbf{g}},\ n \in \mathbb{N}\}$ is an admissible mesh in $K_{\mathbf{g}}$ of cardinality at most $O\left(n^{d+\frac{2d(d-1)}{r}}\right)$.

This verifies the first statement of Theorem 1.

In case when the graph domain is analytic we can use a Bernstein type result (see [10]) which provides in (10) an approximation order $O(\mathrm{e}^{-cm})$ instead of $\frac{c_1}{m^r}$. Then setting $m := A \ln n$ with a proper $A > 0$ and repeating the above argument leads to an optimal admissible mesh of required cardinality in the analytic case. □

Remark 1. The optimal admissible meshes of Theorem 1 provide estimate (1) with some $c_K > 0$. The proof of Theorem 1 can be easily modified so that this constant can be replaced by $1 + \epsilon$ with an arbitrary $\epsilon > 0$. Of course, in this case the constant appearing in the estimate for the cardinality of optimal meshes will depend on $\epsilon > 0$, as well.

There are numerous domains for which it is more natural to use spherical coordinates in $\mathbb{R}^d$. In this case the boundary of the domain is often given by graphs of trigonometric polynomials. This leads to the need of a result similar to estimate (2) for trigonometric polynomials on intervals shorter than the period. Our next proposition accomplishes this goal.

Proposition 3. *Let $U_n,\ n \in \mathbb{N}$ be a sequence of linear subspaces in $C^1[-w, w]$, $w > 0$ such that with some fixed $A, B > 0$ the following Markov and Bernstein type inequalities hold for every $p \in U_n,\ x \in (-w, w)$ and $n \in \mathbb{N}$:*

(i) $\|p'\|_{[-w,w]} \le \dfrac{An^2}{w} \|p\|_{[-w,w]}$;

(ii) $|p'(x)| \le \dfrac{Bn}{\sqrt{w^2 - x^2}} \|p\|_{[-w,w]}$.

Then $[-w, w]$ possesses an optimal admissible mesh for $U_n \mathbf{Y} = \{Y_n \subset [-w, w],\ n \in \mathbb{N}\}$ satisfying

$$\|p\|_{[-w,w]} \le 2\|p\|_{Y_n}, \quad p \in U_n,\ n \in \mathbb{N}$$

with $\operatorname{card}(Y_n) \le \pi \max(\sqrt{A}, 2B) n$.

Proof of Proposition 3. Set $Y_n := \left\{ y_j := w \cos \frac{\pi j}{mn},\ 0 \le j \le mn \right\} \subset [-w, w]$, where m is the smallest integer such that $m > \pi \max(\sqrt{A}, 2B)$. For an arbitrary $p \in U_n$ let $\|p\|_{[-w,w]} = p(x)$, where without loss of generality we may assume that $x \ge 0$.

Case 1. $y_{j+1} \le x \le y_j$ for some $1 \le j \le mn-2$. Then clearly by the mean value theorem and assumption (ii) (using also that $0 \le x \le y_j$, i.e., $\frac{\pi j}{mn} \le \pi/2$)

$$\begin{aligned}|p(x)-p(y_j)| &\le (y_j - y_{j+1})\frac{Bn}{\sqrt{w^2-y_j^2}}\|p\|_{[-w,w]}\\ &\le \frac{Bn\sin\frac{\pi}{2mn}\sin\left(\frac{\pi j}{mn}+\frac{\pi}{2mn}\right)}{\sin\frac{\pi j}{mn}}\|p\|_{[-w,w]}\\ &\le \frac{\pi B}{m}\|p\|_{[-w,w]} \le \frac{1}{2}\|p\|_{[-w,w]}.\end{aligned}$$

Since $\|p\|_{[-w,w]} = p(x)$ we must have $|p(y_j)| \ge \frac{1}{2}\|p\|_{[-w,w]}$. This completes the proof of Case 1.

Case 2. Now let $x > y_1$. Using again the mean value theorem and assumption (i) of the lemma we have

$$\begin{aligned}|p(x)-p(y_1)| &\le \frac{An^2}{w}(w-y_1)\|p\|_{[-w,w]} = An^2\left(1-\cos\frac{\pi}{mn}\right)\|p\|_{[-w,w]}\\ &\le \frac{A\pi^2}{2m^2}\|p\|_{[-w,w]} \le \frac{1}{2}\|p\|_{[-w,w]}.\end{aligned}$$

So similarly to Case 1 we obtain $|p(y_1)| \ge \frac{1}{2}\|p\|_{[-w,w]}$. □

By the classical Markov and Bernstein inequalities Proposition 3 is applicable for univariate algebraic polynomials of degree n on $[-1,1]$ (with $A = B = 1$) and to univariate trigonometric polynomials on $[-\pi,\pi]$. Moreover, it is also known (see [1, E19, p. 242]) that the assumptions of the above proposition hold for trigonometric polynomials of degree n when $w < \pi$ with suitable constants A, B (depending on w).

Let us denote by T^d the space of all real trigonometric polynomials in d variables. Then by induction applied together with Proposition 3 we obtain

Corollary 1. *Let $K \in \mathbb{R}^d$ be a trigonometric polynomial image of the rectangular domain $D = [-w_1, w_1] \times \cdots \times [-w_d, w_d] \in \mathbb{R}^d$, $0 < w_j \le \pi, 1 \le j \le d$, i.e., $K = \mathbf{T}(D)$ with $\mathbf{T} = (t_1, \dots, t_d)$, $t_j \in T^d$, $1 \le j \le d$. Then K possesses an optimal admissible mesh.*

Corollary 1 can be applied to a variety of domains which are not polynomial graph domains, i.e., to which Proposition 1 does not apply.

Example 1. Let $K = \{0 \le z \le 1 - \sqrt{x^2+y^2}, x^2+y^2 \le 1\}$ be the standard circular cone in $\mathbb{R}^3$. Then it is easy to see that $K = \mathbf{T}(D)$ where $D = [0, 2\pi] \times [0, \pi/2] \times [0, \pi/2]$,

$$\mathbf{T} = (\sin\phi\cos\psi, \sin\phi\sin\psi, (1-\sin\phi)\sin\xi), \quad (\psi, \phi, \xi) \in D.$$

Thus Corollary 1 implies that cone K possesses an optimal admissible mesh.

3. Admissible meshes in convex and star like domains

In the preceding section we investigated how to construct optimal or near optimal admissible meshes in domains with certain analytic properties. In this section we choose a different approach: instead of assuming that the underlying domain has certain analytic properties we consider sets with nice geometry, that is convex sets or star like sets with smooth boundary. It is a straightforward consequence of Markov Inequality (see [5]) that any *convex body* $K \subset \mathbb{R}^d$ possesses an admissible mesh of cardinality at most $O(n^{2d})$. Our next result presents an improvement of this bound on the cardinality of admissible meshes. This improvement will be achieved by a combined application of both Bernstein and Markov type inequalities on convex bodies. In addition, using John's maximal ellipsoidal theorem [7] we shall make our estimates domain independent.

Theorem 2. *Any convex body $K \subset \mathbb{R}^d$ possesses and admissible mesh $\mathbf{Y} = \{Y_n \subset K,\ n \in \mathbb{N}\}$ satisfying relation (1) with $\operatorname{card}(Y_n) \le cn^2 \ln n$ if $d = 2$ and $\operatorname{card}(Y_n) \le c_d n^{2d-2}$ when $d > 2$, where c is an absolute constant and c_d depends on d only.*

In what follows we shall say that the compact set $K \subset \mathbb{R}^d$ containing the origin in its interior is *star like* if for every $\mathbf{x} \in K$ we have $[\mathbf{0}, \mathbf{x}] \subset K$. Given a star like set K, consider its Minkowski functional defined as

$$|\mathbf{x}|_K := \inf\{\alpha > 0 : \mathbf{x}/\alpha \in K\}, \quad \mathbf{x} \in \mathbb{R}^d.$$

The star like property of K yields that $K = \{\mathbf{x} : |\mathbf{x}|_K \le 1\}$ and $|t\mathbf{x}|_K = t|\mathbf{x}|_K, \mathbf{x} \in \mathbb{R}^d,\ t > 0$. Moreover, it is well known that when K is a convex body this functional is a norm on $\mathbb{R}^d$. Let us also denote by $S^{d-1} := \{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x}| = 1\}$ the unit sphere in $\mathbb{R}^d$. Moreover, $B(\mathbf{x}, r)$ is the closed ball centered at $\mathbf{x} \in \mathbb{R}^d$ and radius $r > 0$.

Assume now that the star like set K satisfies $B(\mathbf{0}, r) \subset K \subset B(\mathbf{0}, R)$ for some $0 < r \le R$. Then for any $\mathbf{x} \in \mathbb{R}^d$ we have the obvious inclusions $\frac{r\mathbf{x}}{|\mathbf{x}|} \in B(\mathbf{0}, r) \subset K$ and $\frac{\mathbf{x}}{|\mathbf{x}|_K} \in K \subset B(\mathbf{0}, R)$ implying that

$$r|\mathbf{x}|_K \le |\mathbf{x}| \le R|\mathbf{x}|_K, \quad \mathbf{x} \in \mathbb{R}^d. \tag{14}$$

Lemma 1. *Let K be a convex body such that $B(\mathbf{0}, r) \subset K \subset B(\mathbf{0}, R)$ for some $0 < r \le R$. Then for any $\mathbf{x}, \mathbf{y} \in S^{d-1}$ we have*

$$\left| \frac{\mathbf{x}}{|\mathbf{x}|_K} - \frac{\mathbf{y}}{|\mathbf{y}|_K} \right| \le \left(1 + \frac{R}{r}\right) R|\mathbf{x} - \mathbf{y}|.$$

Proof. For any $\mathbf{x}, \mathbf{y} \in S^{d-1}$ using that $|\mathbf{x}|_K$ is a norm we have by (14)

$$\begin{aligned} \left| \frac{\mathbf{x}}{|\mathbf{x}|_K} - \frac{\mathbf{y}}{|\mathbf{y}|_K} \right| &\le \frac{|\mathbf{x}|\,\big||\mathbf{x}|_K - |\mathbf{y}|_K\big|}{|\mathbf{x}|_K |\mathbf{y}|_K} + \frac{|\mathbf{x}|_K |\mathbf{x} - \mathbf{y}|}{|\mathbf{x}|_K |\mathbf{y}|_K} \\ &\le \frac{|\mathbf{x}||\mathbf{x} - \mathbf{y}|_K}{|\mathbf{x}|_K |\mathbf{y}|_K} + \frac{|\mathbf{x} - \mathbf{y}|}{|\mathbf{y}|_K} \le \left(1 + \frac{R}{r}\right) R|\mathbf{x} - \mathbf{y}|, \end{aligned}$$

which is the needed estimate of Lemma 1. $\square$

Remark. Note that in the proof of Lemma 1 we only used the semi-additivity of the Minkowski functional, i.e., the fact that the functional $|\mathbf{x}|_K$ is Lip 1. This is an immediate consequence of K being convex.

In what follows for the sets A, B in $\mathbb{R}^d$ we shall denote by

$$d(A, B) := \sup_{\mathbf{x} \in A} \inf_{\mathbf{y} \in B} |\mathbf{x} - \mathbf{y}|$$

the density of the set B in the set A.

Lemma 1 yields the next corollary on the density of discrete point sets on the boundary of convex bodies.

Corollary 2. *If a convex body K satisfies conditions of Lemma 1 then for any $\delta > 0$ we can choose a discrete set Y_n on its boundary ∂K satisfying*

$$d(\partial K, Y_n) \leq \left(1 + \frac{R}{r}\right) R\delta \tag{15}$$

so that

$$\mathrm{card}(Y_n) \leq c_1(d)\delta^{1-d} \tag{16}$$

with some constant $c_1(d) > 0$ depending only on d.

Proof. It is known that for $K = S^{d-1}$ the above corollary holds with δ on the right-hand side of (15). Thus in view of Lemma 1 the needed statement easily follows.

Proof of Theorem 2. We start by pointing out how to obtain domain independent estimates for the convex body K. First it should be noted that the statement of Theorem 2 is invariant under affine transformations. Second, by the John maximal ellipsoidal theorem [7] there exists a unique ellipsoidal E_K of maximal volume and center $\mathbf{c}_K$ such that $E_K \subset K \subset \mathbf{c}_K + d(E_K - \mathbf{c}_K)$ where $E_K = T(B(\mathbf{0}, 1))$ with some affine map $T : \mathbb{R}^d \to \mathbb{R}^d$. Thus we may assume without loss of generality that $B(\mathbf{0}, 1) \subset K \subset B(\mathbf{0}, d)$. We shall denote by $c_j(d)$ positive constants depending only on d.

Now set for any $1 \leq j \leq n,\ n \in \mathbb{N}$

$$\rho_j := \cos\frac{aj\pi}{n}, \qquad h_j := \frac{a\sin\frac{aj\pi}{n}}{n}; \qquad K_j := \{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x}|_K \leq \rho_j\}, \tag{17}$$

where $a = \frac{1}{12d^2}$. (Basically, we need the constant a in (17) to be sufficiently small, but in order to obtain domain independent estimate we shall make this explicit choice. The same holds with respect to the constant A to be chosen a few lines below.) Clearly, $B(\mathbf{0}, \rho_j) \subset K_j \subset B(\mathbf{0}, \rho_j d)$, $1 \leq j \leq n$. Thus applying Corollary 2 to K_j with $\delta := \frac{h_j}{d(d+1)}$, $r = \rho_j$ and $R = \rho_j d$ it follows that we can chose $N_j \leq c_1(d)(1+d)^{2(d-1)} h_j^{1-d} \leq c_2(d) h_j^{1-d}$ points $\{y_{i,j} \in \partial K_j,\ 1 \leq i \leq N_j\} := Y_{n,j}$ on ∂K_j so that

$$d(\partial K_j, Y_{n,j}) \leq h_j, \quad 1 \leq j \leq n. \tag{18}$$

Clearly setting $A := \frac{1}{24d^4}$ we can choose a discrete set $Y_n^* \subset K_n$ so that

$$d(K_n, Y_n^*) \leq \frac{A}{n}, \tag{19}$$

and $\mathrm{card}(Y_n^*) \leq c_3(d)n^d$.

Set now $Y_n := \cup_{1 \leq j \leq n} Y_{n,j} \cup Y_n^*$. The cardinality of this set can be estimated as follows

$$\begin{aligned}\mathrm{card}(Y_n) &\leq c_3(d)n^d + \sum_{j=1}^n N_j \leq c_4(d)n^d\left(1 + \frac{1}{n}\sum_{j=1}^n \frac{1}{\sin^{d-1}\frac{aj\pi}{n}}\right)\\ &\leq c_5(d)n^d\left(1 + n^{d-2}\sum_{j=1}^n \frac{1}{j^{d-1}}\right).\end{aligned}$$

When $d = 2$ the above estimate yields $\mathrm{card}(Y_n) \le cn^2 \ln n$ with an absolute constant $c > 0$. For $d > 2$ it clearly yields $\mathrm{card}(Y_n) \le c_6(d) n^{2d-2} m$, i.e., the set Y_n has the required cardinality.

It remains to show now that Y_n is an admissible mesh. Consider an arbitrary $p \in P_n^d$ with $\|p\|_K = p(\mathbf{x}), \mathbf{x} \in K$. Clearly, $|\mathbf{x}|_K = \rho$ for some $0 \le \rho \le 1$. From now on we shall distinguish between three cases depending on the size of ρ.

Case 1. $\rho_n \le \rho \le \rho_1$. Then $\rho_{j+1} \le \rho \le \rho_j$ for some $1 \le j \le n-1$. Hence by (17) easy calculations yield

$$\rho_j - \rho \le \rho_j - \rho_{j+1} = \cos \frac{aj\pi}{n} - \cos \frac{a(j+1)\pi}{n} \le \frac{2a}{n} \sin \frac{aj\pi}{n} = 2h_j. \tag{20}$$

Let $\mathbf{z}^* \in K_j$ be such that $\mathbf{z}^* = t\mathbf{x}$ for some $t \ge 1$. Clearly, $t|\mathbf{x}|_K = \rho_j$. Then by (20)

$$|\mathbf{x} - \mathbf{z}^*|_K = |\mathbf{x}|_K (t-1) = \rho_j - |\mathbf{x}|_K = \rho_j - \rho \le 2h_j. \tag{21}$$

Furthermore, by (18) for $\mathbf{z}^* \in K_j$ there exists $\mathbf{y} \in Y_{n,j}$ such that $|\mathbf{z}^* - \mathbf{y}| \le h_j$. This and (21) yield

$$|\mathbf{x} - \mathbf{y}| \le 3h_j, \quad \mathbf{y} \in Y_{n,j}. \tag{22}$$

Now in order to proceed we shall need a Bernstein type inequality for convex bodies proved in [11]: given any $p \in P_n^d$, $\mathbf{w} \in S^{d-1}$ and $\mathbf{z} \in K$ we have

$$|D_{\mathbf{w}} p(\mathbf{z})| \le \frac{\sqrt{2} n \|p\|_K}{\Delta_K(\mathbf{z})}, \tag{23}$$

where $D_{\mathbf{w}}$ stands for the derivative in the direction $\mathbf{w}$ and

$$\Delta_K(\mathbf{z}) := 2 \inf \frac{\sqrt{|\mathbf{z} - \mathbf{a}||\mathbf{z} - \mathbf{b}|}}{|\mathbf{a} - \mathbf{b}|}$$

with the inf above being taken over all $\mathbf{a}, \mathbf{b} \in \partial K$ such that $\mathbf{z} \in [\mathbf{a}, \mathbf{b}]$. Another important related statement proved in [11] says that this inf can be attained only at points $\mathbf{a}, \mathbf{b} \in \partial K$ where K possesses parallel supporting hyperplanes. So if this inf is attained for given $\mathbf{a}, \mathbf{b} \in \partial K$ then it follows from $B(\mathbf{0}, 1) \subset K$ that $|\mathbf{a} - \mathbf{b}| \ge 2$. Thus without loss of generality we may assume that $|\mathbf{z} - \mathbf{b}| \ge 1$ yielding by $K \subset B(\mathbf{0}, d)$ that

$$\Delta_K(\mathbf{z}) \ge \frac{\sqrt{|\mathbf{z} - \mathbf{a}|}}{d} \ge \frac{\sqrt{\mathrm{dist}(\mathbf{z}, \partial K)}}{d}. \tag{24}$$

Furthermore, since $B(\mathbf{0}, 1) \subset K \subset B(\mathbf{0}, d)$ it is easy to see that

$$\mathrm{dist}(\mathbf{z}, \partial K) \ge \frac{1}{d}(1 - |\mathbf{z}|_K).$$

Hence and by (24)

$$\Delta_K(\mathbf{z}) \ge \frac{\sqrt{1 - |\mathbf{z}|_K}}{d^2}. \tag{25}$$

Mean value theorem, (22) and (23) imply that with some $\mathbf{z} \in [\mathbf{x}, \mathbf{y}]$

$$|p(\mathbf{x}) - p(\mathbf{y})| \le h_j \frac{3\sqrt{2} n \|p\|_K}{\Delta_K(\mathbf{z})}.$$

Since $\mathbf{x}, \mathbf{y} \in K_j$ it follows by convexity that $\mathbf{z} \in K_j$, as well, that is $|\mathbf{z}|_K \leq \rho_j$. Using the last estimate together with (25) yields

$$|p(\mathbf{x}) - p(\mathbf{y})| \leq 3\sqrt{2}d^2 \frac{nh_j \|p\|_K}{\sqrt{1-\rho_j}}. \tag{26}$$

Note that by (17) $nh_j = a\sqrt{1-\rho_j^2}$. Using this relation in (26) and recalling that $a = \frac{1}{12d^2}$ we arrive at

$$|p(\mathbf{x}) - p(\mathbf{y})| \leq 3\sqrt{2}d^2 a \|p\|_K \sqrt{1+\rho_j} \leq 6d^2 a\|p\|_K \leq \frac{1}{2}\|p\|_K.$$

Since $\|p\|_K = p(\mathbf{x})$ and $\mathbf{y} \in Y_{n,j} \subset Y_n$ it follows from the last estimate that $\|p\|_K \leq 2\|p\|_{Y_n}$, i.e. Y_n is an admissible mesh in this case.

Case 2. $\rho_1 \leq \rho \leq 1$. Then $\mathbf{z} := t\mathbf{x} \in \partial K_1$ with $t = \rho_1/\rho$. Hence by (18) there exists $\mathbf{y} \in Y_{n,1} \subset Y_n$ such that $|\mathbf{z} - \mathbf{y}| \leq h_1$. In addition, using (14) with $R = d$ yields

$$\begin{aligned} |\mathbf{z} - \mathbf{x}| &\leq |\mathbf{x}|(1-\rho_1/\rho) \leq d|\mathbf{x}|_K(1-\rho_1/\rho) = d(\rho - \rho_1) \\ &\leq d(1-\rho_1) = d\left(1 - \cos\frac{a\pi}{n}\right) \leq \frac{da^2\pi^2}{2n^2}. \end{aligned}$$

Thus recalling (17)

$$|\mathbf{x} - \mathbf{y}| \leq h_1 + \frac{\mathrm{d}a^2\pi^2}{2n^2} \leq \frac{\mathrm{d}a^2\pi^2}{n^2}. \tag{27}$$

Now we need a Markov type inequality verified by Wilhelmsen [16]: for any $\mathbf{w} \in S^{d-1}$

$$\|D_{\mathbf{w}} p\|_K \leq \frac{2n^2\|p\|_K}{\Omega}, \tag{28}$$

where Ω is the radius of the largest ball inscribed into convex body K. Since $\Omega = 1$ in our case (27) and (28) yield

$$|p(\mathbf{x}) - p(\mathbf{y})| \leq 2\mathrm{d}a^2\pi^2\|p\|_K \leq \frac{1}{2}\|p\|_K.$$

Obviously, this yields $\|p\|_K \leq 2\|p\|_{Y_n}$.

Case 3. $0 \leq \rho \leq \rho_n = \cos a\pi$. In this case $\mathbf{x} \in K_n$ hence by (19) there exists $\mathbf{y} \in Y_n^* \subset K_n$ such that $|\mathbf{x} - \mathbf{y}| \leq A/n$ holds. The mean value theorem and (23) imply that with some $\mathbf{z} \in K_n$

$$|p(\mathbf{x}) - p(\mathbf{y})| \leq \frac{A\sqrt{2}\|p\|_K}{\Delta_K(\mathbf{z})}.$$

Thus using again (25) and recalling that $a = \frac{1}{12d^2}$, $A = \frac{1}{24d^4}$ we arrive at

$$\begin{aligned} |p(\mathbf{x}) - p(\mathbf{y})| &\leq \frac{Ad^2\sqrt{2}\|p\|_K}{\sqrt{1-|\mathbf{z}|_K}} \leq \frac{Ad^2\sqrt{2}\|p\|_K}{\sqrt{1-\rho_n}} = \frac{Ad^2\sqrt{2}\|p\|_K}{\sqrt{1-\cos a\pi}} \\ &\leq \frac{Ad^2}{a}\|p\|_K \leq \frac{\|p\|_K}{2}. \end{aligned}$$

Again this yields that $\|p\|_K \leq 2\|p\|_{Y_n}$, i.e. Y_n is an admissible mesh in this case, too. $\square$

In case when $d = 2$ Theorem 2 provides an estimate of near optimal order $n^2 \ln n$ which differs from the optimal only by an $\ln n$ factor. When $d > 2$ the estimate $O(n^{2d-2})$ is less satisfactory, but it still improves the bound n^{2d} given in [5]. In addition, it should be noted that the constants in Theorem 2 are domain independent.

It seems to be plausible that any convex body possesses optimal admissible meshes. Thus we would like to formulate the next

Conjecture. *Any convex body $K \subset \mathbb{R}^d$, $d \geq 2$ possesses an optimal admissible mesh.*

We shall now verify the above conjecture in the special case when K is a polytope. In order to formulate the corresponding result let us introduce an auxiliary notation. If K is a convex polytope denote by $F_j(l)$ its l-dimensional faces, $0 \leq l \leq d$. Now for any l-dimensional face $F_j(l)$ we recursively define the quantity $s(F_j(l))$, $2 \leq l \leq d$ as follows. When $d = 2$, $s(F_j(2))$ equals the number of vertices of the two-dimensional polygon $F_j(2)$. For any $2 < l \leq d$ put $s(F_j(l)) = \sum s(F_{k,j}(l-1))$ where the sum is taken over all $l-1$-dimensional faces of the l-dimensional polytope $F_j(l)$. Clearly, when $l = d$ we have $F_j(d) = K$, i.e., the above definition introduces a quantity $s(F_j(d)) := s(K)$ which provides a certain count of the vertices of K. When $d = 2$ the quantity $s(K)$ is just the total number of vertices of the polygon.

Theorem 3. *Any convex polytope $K \subset \mathbb{R}^d$, $d \geq 1$ possesses an optimal admissible mesh Y_n satisfying (1) with $c_1 = 2^d$ and such that* $\operatorname{card}(Y_n) \leq s(K)([n\pi/2]+1)^d$, $n \in \mathbb{N}$.

Proof of Theorem 3. We may assume that $\mathbf{0}$ is in the interior of K. We are going to prove the theorem by induction on d.

When $d = 1$ that is K is a line segment and $s(K) = 2$ the statement follows immediately by (2).

Assume now that the above claim holds for $d-1$. Let $F_j(d-1)$, $1 \leq j \leq N$ be the $d-1$-dimensional faces of the polytope $K \subset \mathbb{R}^d$, and set $m_n := [n\pi/2]+1$, $t_i := \cos \frac{\pi i}{m_n}$, $0 \leq i \leq m_n$,

$$F_{i,j} := \left\{ \frac{t_i+1}{2}\mathbf{x}, \mathbf{x} \in F_j(d-1) \right\}, \qquad F_j^* := \left\{ \frac{t+1}{2}\mathbf{x}, \mathbf{x} \in F_j(d-1), |t| \leq 1 \right\},$$
$$1 \leq j \leq N.$$

Since each $F_{i,j}$ is a $d-1$-dimensional polytope with $s(F_{i,j}) = s(F_j(d-1))$, $0 \leq i \leq m_n$, it follows by the induction hypothesis that each $F_{i,j}$, $0 \leq i \leq m_n$ possesses an admissible mesh $\mathbf{Y}_{i,j} = \{Y_{n,i,j} \subset F_{i,j},\ n \in \mathbb{N}\}$ such that for every $n \in \mathbb{N}$, $0 \leq i \leq m_n$ and $1 \leq j \leq N$ we have

$$\|p\|_{F_{i,j}} \leq 2^{d-1} \|p\|_{Y_{n,i,j}}, \quad p \in P_n^d, \tag{29}$$

and

$$\operatorname{card}(Y_{n,i,j}) \leq s(F_j(d-1))(m_n)^{d-1}. \tag{30}$$

Consider any $\mathbf{x} \in F_j(d-1)$, $p \in P_n^d$ and set $g(u) := p(\frac{\cos u+1}{2}\mathbf{x})$. Then by (29)

$$\left| g\left(\frac{\pi i}{m_n}\right) \right| = \left| p\left(\frac{t_i+1}{2}\mathbf{x}\right) \right| \leq \|p\|_{F_{i,j}} \leq 2^{d-1}\|p\|_{Y_{n,i,j}}, \quad 0 \leq i \leq m_n. \tag{31}$$

Note that $g \in T_n^1$ is an even trigonometric polynomial of degree at most n and therefore by the Bernstein inequality

$$\|g'\|_{[-\pi,\pi]} \leq n\|g\|_{[-\pi,\pi]}.$$

Moreover, if $\|g\|_{[-\pi,\pi]} = g(u_0)$, $u_0 \in [0,\pi]$ we can find $\frac{\pi i}{m_n}$, $0 \le i \le m_n$ so that $\left|u_0 - \frac{\pi i}{m_n}\right| \le \frac{\pi}{2m_n}$. Since we have in addition $g'(u_0) = 0$ it follows by $m_n > n$ and repeated application of the Bernstein inequality

$$\left|g(u_0) - g\left(\frac{\pi i}{m_n}\right)\right| \le \frac{\left(u_0 - \frac{\pi i}{m_n}\right)^2}{2}\|g''\|_{[-\pi,\pi]} \le \frac{\|g\|_{[-\pi,\pi]}}{2},$$

i.e., by (31) for any $\mathbf{y} \in F_j^*$, $1 \le j \le N$, $p \in P_n^d$

$$|p(\mathbf{y})| \le \|g\|_{[-\pi,\pi]} \le 2\left|g\left(\frac{\pi i}{m_n}\right)\right| \le 2^d \|p\|_{Y_{n,i,j}}, \quad 0 \le i \le m_n.$$

Clearly, this last estimate yields that $Y_n := \cup_{1\le j\le N, 0\le i\le m_n} Y_{n,i,j}$ is an admissible mesh for K for which (1) holds with $c_1 = 2^d$. Moreover, by (30)

$$\operatorname{card}(Y_n) \le \sum_{1\le j\le N, 0\le i\le m_n} \operatorname{card}(Y_{n,i,j}) \le (m_n)^d \sum_{1\le j\le N} s(F_j(d-1)) = s(K)(m_n)^d. \quad \square$$

Now we shall replace the condition of convexity by the more general star like property. We shall also assume that the boundary ∂K of K is *smooth*, that is its Minkowski functional $|\mathbf{x}|_K$ is differentiable on $\mathbb{R}^d \setminus \{\mathbf{0}\}$. Denote by $\partial|\mathbf{x}|_K$ the gradient of the Minkowski functional, and let us say that the domain K is $C^{1+\alpha}$ if $\partial|\mathbf{x}|_K$ is Lipα, $0 < \alpha \le 1$ on compact subsets of $\mathbb{R}^d \setminus \{\mathbf{0}\}$.

Theorem 4. *Let $K \subset \mathbb{R}^d, d \ge 2$ be a star like $C^{1+\alpha}$ domain, $0 < \alpha \le 1$. Then K possesses an admissible mesh $\mathbf{Y} = \{Y_n \subset K,\ n \in \mathbb{N}\}$ satisfying relation* (1) *with* $\operatorname{card}(Y_n) = O\left(n^{\frac{2d+\alpha-1}{\alpha+1}}\right)$, $n \in \mathbb{N}$. *In particular, if K is C^2 (i.e., $\alpha = 1$) then* $\operatorname{card}(Y_n) = O(n^d)$, $n \in \mathbb{N}$ *and hence K possesses an optimal admissible mesh.*

Clearly the above theorem yields the conjecture on the existence of optimal admissible meshes for convex domains in the case of convex bodies with C^2 boundary. The proof of Theorem 4 will rely on a tangential Markov type inequality proved in [10]. It is easy to see that when K is smooth $\partial|\mathbf{x}|_K$ provides the tangent direction to ∂K at $\mathbf{x} \in \partial K$. Now for any $\mathbf{x} \in \partial K$ denote by $T(\mathbf{x})$ the set of all unit tangent vectors to ∂K at $\mathbf{x} \in \partial K$.

Consider the *tangential Markov Factor* of K given by

$$M_n(K) := \sup\{|D_{\mathbf{w}} p(\mathbf{x})| : p \in P_n^d, \|p\|_K \le 1, \mathbf{x} \in \partial K, \mathbf{w} \in T(\mathbf{x})\}.$$

This Markov Factor gives the size of tangential derivatives of polynomials on the boundary of K. It is shown in [10] that whenever $K \subset \mathbb{R}^d$, $d \ge 2$ is a star like $C^{1+\alpha}$ domain

$$M_n(K) = O\left(n^{\frac{2}{1+\alpha}}\right).$$

It is a straightforward consequence of this estimate that for any $p \in P_n^d$ with $\|p\|_K \le 1$, $\mathbf{x} \in K$, $\mathbf{w} \in T(\mathbf{x}/|\mathbf{x}|_K)$

$$|D_{\mathbf{w}} p(\mathbf{x})| \le c_K \left(n^{\frac{2}{1+\alpha}}\right) \tag{32}$$

with a constant $c_K > 0$ depending only on K.

Proof of Theorem 4. Since K has nonempty interior for some $0 < r < R$ we have $B(\mathbf{0}, 2r) \subset K \subset B(\mathbf{0}, R)$ and hence relation (14) holds.

For a given constant $A > 0$ to be chosen below, depending only on K and d set

$$N := \left[An \cos^{-1} \frac{r}{R}\right] + 1, \qquad \rho_j := \cos \frac{j}{An}, \qquad K_j := \{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x}|_K \le \rho_j\},$$
$$1 \le j \le N. \tag{33}$$

Then clearly $\rho_N \le \frac{r}{R}$ hence it follows by (14) that for any $\mathbf{x} \in K_N, |\mathbf{x}| \le R|\mathbf{x}|_K \le r$, i.e., $K_N \subset B(\mathbf{0}, r)$.

Next we need to recall that Lemma 1 proved for a convex body K. In the proof of this lemma we used inequality $|\mathbf{x}|_K - |\mathbf{y}|_K \le |\mathbf{x} - \mathbf{y}|_K$, which followed from the convexity of K. In fact this was the only place were convexity was applied. Since in our case K is $C^{1+\alpha}$ on compact domains separated away from the origin, in particular it follows that the functional $|\mathbf{x}|_K$ is Lip 1. Thus Lemma 1 is applicable to K_j, as well, with some constant depending only on the Lipschitz constant of K_j appearing on the right-hand side of the estimate of the lemma. Since $1 \ge \rho_j \ge \cos \frac{1}{A} > 0, 1 \le j \le N$, it follows that Lemma 1 holds for each $K_j, 1 \le j \le N$ with some constant depending on K only. This and Corollary 2 yields that for any $h > 0$ we can choose finite point sets $Y_j \subset \partial K_j, 1 \le j \le N$ satisfying

$$d(\partial K_j, Y_j) \le h \tag{34}$$

so that with some constant $c_0 > 0$ depending only on K and d

$$\operatorname{card}(Y_j) \le c_0 h^{1-d}, \quad 1 \le j \le N. \tag{35}$$

(From now on we shall denote by c_j positive constants depending only on K and d.) In addition, we can choose $Y_n^* \subset B(\mathbf{0}, r)$ so that

$$\operatorname{card}(Y_n^*) \le c_1 n^d, \qquad d(B(\mathbf{0}, r), Y_n^*) \le \frac{1}{An}. \tag{36}$$

Now set $h := \left(\frac{1}{An}\right)^{\frac{2}{1+\alpha}}$, $Y_n := \cup_{1 \le j \le N} Y_j \cup Y_n^*$. Using (35) and (36) we have

$$\operatorname{card}(Y_n) \le c_0 N h^{1-d} + c_1 n^d \le c_0 \pi (An)^{\frac{2d+\alpha-1}{\alpha+1}} + c_1 n^d \le c_2 n^{\frac{2d+\alpha-1}{\alpha+1}}.$$

Thus the mesh Y_n has the required cardinality.

It remains to show now that Y_n is an admissible mesh. Consider any $p \in P_n^d, p(\mathbf{x}) = \|p\|_K, \mathbf{x} \in K$. Again we shall distinguish three cases.

Case 1. $\rho_N \le |\mathbf{x}|_K \le \rho_1$. Then we can find $1 \le j \le N - 1$ so that $\rho_{j+1} \le |\mathbf{x}|_K \le \rho_j$. Set $g(t) := p(t\mathbf{x}/|\mathbf{x}|_K), 0 \le t \le 1, g \in P_n^1, \|g\|_{[0,1]} \le \|p\|_K$. Then by the Bernstein inequality

$$|g(\rho_j) - g(|\mathbf{x}|_K)| \le \frac{n\|g\|_{[0,1]}}{\sqrt{\xi(1-\xi)}}(\rho_j - \rho_{j+1})$$

with some $|\mathbf{x}|_K \le \xi \le \rho_j$. Clearly $\xi \ge |\mathbf{x}|_K \ge \rho_N \ge \frac{r}{R} + O(1/n)$, i.e., above the estimate yields

$$|g(\rho_j) - g(|\mathbf{x}|_K)| \le c_3 \frac{n\|p\|_K}{\sqrt{1-\rho_j}}(\rho_j - \rho_{j+1}). \tag{37}$$

Since $\rho_j := \cos\frac{j}{An}$ a simple calculation yields

$$\rho_j - \rho_{j+1} \le \frac{c_4}{An}\sin\frac{j}{An} = \frac{c_4}{An}\sqrt{1-\rho_j^2} \le \frac{2c_4}{An}\sqrt{1-\rho_j}.$$

Using this estimate in (37) we obtain

$$|g(\rho_j) - g(|\mathbf{x}|_K)| \le \frac{2c_3c_4}{A}\|p\|_K \le \frac{1}{2}\|p\|_K$$

provided that $A > 4c_3c_4$. Note that $g(|\mathbf{x}|_K) = p(\mathbf{x}) = \|p\|_K$, i.e., the last estimate implies

$$|p(\rho_j\mathbf{x}/|\mathbf{x}|_K)| = |g(\rho_j)| \ge \frac{\|p\|_K}{2}. \tag{38}$$

Furthermore, by (34) for the given $\rho_j\mathbf{x}/|\mathbf{x}|_K \in \partial K_j$ there exists an $\mathbf{y} \in Y_j$ such that

$$|\rho_j\mathbf{x}/|\mathbf{x}|_K - \mathbf{y}| \le h = \left(\frac{1}{An}\right)^{\frac{2}{1+\alpha}}. \tag{39}$$

From now on we shall work in the two-dimensional plane spanned by $\mathbf{0}$, $\mathbf{x}$, $\mathbf{y}$. So without loss of generality assume that this plane is $\mathbb{R}^2$ and

$$\partial K \cap \mathbb{R}^2 = \Gamma := \{(\gamma(\phi)\cos\phi, \gamma(\phi)\sin\phi, 0, \ldots, 0), \phi \in [-\pi, \pi]\}$$

with some 2π-periodic $C^{1+\alpha}$ function $\gamma(\phi)$, $\phi \in [-\pi, \pi]$. Then for some $\phi_1, \phi_2 \in [-\pi, \pi]$ we have $\mathbf{x}/|\mathbf{x}|_K = \Gamma(\phi_1)$, $\mathbf{y}/|\mathbf{y}|_K = \Gamma(\phi_2)$. Now set $Q(\phi) := p(\rho_j\Gamma(\phi))$. Then by (38)

$$|Q(\phi_1)| = |p(\rho_j\mathbf{x}/|\mathbf{x}|_K)| \ge \frac{\|p\|_K}{2}, \qquad Q(\phi_2) = p(\rho_j\mathbf{y}/|\mathbf{y}|_K) = p(\mathbf{y}). \tag{40}$$

Moreover, with some $\xi \in [\phi_1, \phi_2]$

$$|Q(\phi_1) - Q(\phi_2)| \le |Q'(\xi)||\phi_1 - \phi_2|, \tag{41}$$

where by (32) using that Γ' is tangent to ∂K

$$|Q'(\xi)| \le \rho_j|\langle\partial p, \Gamma'\rangle|(\xi) \le c_5 n^{\frac{2}{1+\alpha}}\|p\|_K. \tag{42}$$

In addition,

$$|\Gamma(\phi_1) - \Gamma(\phi_2)| \ge \inf_{\phi\in[-\pi,\pi]}|\Gamma'(\phi)||\phi_1 - \phi_2| \ge r|\phi_1 - \phi_2|. \tag{43}$$

Thus by (39)

$$\left(\frac{1}{An}\right)^{\frac{2}{1+\alpha}} \ge |\rho_j\mathbf{x}/|\mathbf{x}|_K - \mathbf{y}| = \rho_j|\Gamma(\phi_1) - \Gamma(\phi_2)| \ge r\rho_N|\phi_1 - \phi_2| \ge c_6|\phi_1 - \phi_2|.$$

Using this last estimate together with (40)–(42) we obtain

$$\frac{\|p\|_K}{2} - |p(\mathbf{y})| \le |Q(\phi_1) - Q(\phi_2)| \le \frac{c_5}{c_6}n^{\frac{2}{1+\alpha}}\left(\frac{1}{An}\right)^{\frac{2}{1+\alpha}}\|p\|_K \le \frac{1}{4}\|p\|_K,$$

provided that $A > \left(\frac{4c_5}{c_6}\right)^{(1+\alpha)/2}$. Hence with a proper choice of A

$$\|p\|_K \le 4|p(\mathbf{y})| \le 4\|p\|_{Y_n}.$$

Case 2. $\rho_1 \le |\mathbf{x}|_K \le 1$. Then setting again $g(t) := p(t\mathbf{x}/|\mathbf{x}|_K),\ t \in [0,1],\ g \in P_n^1$ we have $g(|\mathbf{x}|_K) = p(\mathbf{x}) = \|p\|_K, g(\rho_1) = p(\rho_1\mathbf{x}/|\mathbf{x}|_K), \|g\|_{[0,1]} = \|p\|_K$. Thus using the univariate Markov inequality

$$\|p\|_K - |p(\rho_1\mathbf{x}/|\mathbf{x}|_K)| \le |g(|\mathbf{x}|_K) - g(\rho_1)| \le 2n^2(1-\rho_1)\|p\|_K \le \frac{\|p\|_K}{A^2} \le \frac{\|p\|_K}{2}$$

provided that $A > \sqrt{2}$. Thus $|p(\rho_1\mathbf{x}/|\mathbf{x}|_K)| \ge \frac{\|p\|_K}{2}$. Note that this is the same as (38) for $j = 1$. Now the rest of the proof is analogous to the part of Case 1 which followed after estimate (38).

Case 3. $|\mathbf{x}|_K \le \rho_N$. Recalling that $\rho_N \le \frac{r}{R}$ and using again (14) we have $|\mathbf{x}| \le r$, i.e., $\mathbf{x} \in B(\mathbf{0}, r)$. Clearly by (36) we can find $\mathbf{y} \in Y_n^* \subset B(\mathbf{0}, r)$ for which $|\mathbf{x} - \mathbf{y}| \le \frac{1}{An}$ holds. Since $B(\mathbf{0}, 2r) \subset K$ it follows that we can use the Bernstein type estimate (23) on the ball $B(\mathbf{0}, r)$ yielding that

$$\|D_{\mathbf{w}} p\|_{B(\mathbf{0},r)} \le c_7 n \|p\|_{B(\mathbf{0},2r)}, \quad \mathbf{w} \in S^{d-1}.$$

Therefore

$$|p(\mathbf{x}) - p(\mathbf{y})| \le c_7 n \|p\|_{B(\mathbf{0},2r)} |\mathbf{x} - \mathbf{y}| \le \frac{c_7}{A}\|p\|_K \le \frac{1}{2}\|p\|_K$$

provided that $A > 2c_7$. Since $p(\mathbf{x}) = \|p\|_K$ the above estimated yields $\|p\|_K \le 2\|p\|_{Y_n}$.

This completes the proof of our claim that $\mathbf{Y} = \{Y_n,\ n \in \mathbb{N}\}$ is an admissible mesh. □

4. Bernstein–Markov inequalities and existence of admissible meshes

In the preceding sections two principally different approaches to admissible meshes were discussed. In Section 1 we considered sets with certain analytic properties, namely graph domains bounded by graphs of polynomial, differentiable or analytic functions. In the second section we studied sets with certain geometric properties (convex or star like sets) with the main tools being Bernstein–Markov type inequalities (23), (28) and (38). Bernstein–Markov type inequalities were also used in order to obtain admissible meshes in [2–5]. In particular, as it was pointed out in [5] if the compact domain $K \in \mathbb{R}^d$ satisfies the Markov type inequality

$$\|\partial p\|_K \le c n^r \|p\|_K, \quad p \in P_n^d \tag{44}$$

with some exponent r and constant $c > 0$ depending on K then K possesses an admissible mesh of cardinality $O(n^{rd})$. This can be verified by choosing a uniformly distributed mesh in K with spacing $O(n^{-r})$.

It turns out that even though Bernstein–Markov type inequalities are very useful in finding upper bounds for cardinality of admissible meshes there exist sets which do not have good Markov properties but still possess admissible meshes which are optimal or near optimal. In other words the existence of Bernstein–Markov type inequalities is only a sufficient but not necessary condition for a given compact set to have an admissible mesh of low cardinality. We now present two examples which substantiate this claim.

Example 2. Consider the set $K_r := \{(x, y) \in \mathbb{R}^2 : 0 \le x \le 1; 0 \le y \le x^r\},\ r \in \mathbb{N}$. It is well known (see e.g. [13, Example 1]) that $2r$ is the best possible exponent in Markov type inequality (44) for this set. Using this Markov inequality with exponent $2r$ as mentioned above one can easily get admissible meshes of cardinality $O(n^{2rd})$. However, since K_r is a polynomial graph domain by Proposition 1 it possesses an *optimal* admissible mesh of cardinality $O(n^2)$ for every $r \in \mathbb{N}$. Thus application of the Markov inequality does not lead to sharp results for this domain.

We can explore this idea further by constructing domains on which Markov inequality (44) fails for *any exponent r* but which possess admissible meshes of low cardinality.

Example 3. Consider the set $K = K_1 \cup K_2 \in \mathbb{R}^2$ where

$$K_1 := \{(x, y) \in \mathbb{R}^2 : 0 \le x \le 1/2; 0 \le y \le \mathrm{e}^{-1/x}\},$$
$$K_2 := \{(x, y) \in \mathbb{R}^2 : 1/2 \le x \le 1, 0 \le y \le \mathrm{e}^{-2}\}.$$

It is shown in [13], Example 3 that for some $p \in P_n^2$ with $\|p\|_K = 1$ we have $|\partial p(\mathbf{0})| \ge \mathrm{e}^{cn^{2/3}}$ that is Markov inequality (44) does not hold on K for any exponent r.

We are going to show now that K possesses an admissible mesh of cardinality $O(n^3)$.

Set $x_j := \frac{j}{8n^2}$, $I_j := \{(x_j, y) : (x_j, y) \in K\}$, $1 \le j \le 8n^2$. Then by (2) each interval I_j, $1 \le j \le 8n^2$ possesses a discrete point set $Y_j \subset I_j$ such that $\mathrm{card}(Y_j) \le \pi n$ and for every $p \in P_n^2$ we have

$$\|p\|_{I_j} \le 2\|p\|_{Y_j}, \quad 1 \le j \le 8n^2. \tag{45}$$

Consider now the mesh $Y_n' := \cup_{1 \le j \le 8n^2} Y_j$, $n \in \mathbb{N}$. Obviously, we have

$$\mathrm{card}(Y_n') \le 8\pi n^3.$$

It remains now to verify that (1) holds for the mesh Y_n' with some $c_1 > 0$.

Let $p \in P_n^2$, $\|p\|_K = p(a, b)$, $(a, b) \in K$, $0 \le a \le 1$, $0 \le b \le \mathrm{e}^{-2}$. Set $g(x) := p(x, b) \in P_n^1$. Clearly, $|g(x)| \le \|p\|_K = g(a)$ whenever $1/\ln\frac{1}{b} \le x \le 1$. Moreover, there exists $1 \le j \le 8n^2$ so that $a \le x_j \le a + \frac{1}{8n^2}$. Hence by the classical univariate Markov inequality applied to $g \in P_n^1$ on the interval $\left[1/\ln\frac{1}{b}, 1\right]$ of length $\ge 1/2$

$$|g(a) - g(x_j)| \le \frac{2n^2}{1 + \frac{1}{\ln b}}\|p\|_K(x_j - a) \le 4n^2\|p\|_K(x_j - a) \le \frac{1}{2}\|p\|_K = \frac{1}{2}g(a).$$

Using the last estimate and (45)

$$\|p\|_K = g(a) \le 2|g(x_j)| \le 2\|p\|_{I_j} \le 4\|p\|_{Y_j} \le 4\|p\|_{Y_n'}$$

which is the required estimate.

Thus Y_n' is an admissible mesh in K of cardinality $O(n^3)$.

5. Conclusions

Summarizing the results of this paper we can conclude that when the compact set K satisfies certain analytic or geometric properties we can ensure the existence of admissible meshes in K of low cardinality, or even optimal admissible meshes. This raises the natural question of converse results. That is: does the existence of admissible meshes of low cardinality or optimal admissible meshes in K yield certain structural properties of the domain K?

As it follows from Examples 2 and 3 the answer to this question is not related to Markov type inequalities.

The most natural open question in this respect is the following

Problem. Find a compact set $K \subset \mathbb{R}^d$, $d \ge 1$ which *does not* possess an optimal admissible mesh.

It is plausible that such compact sets with no optimal meshes exist but the above question seems to be unsettled even in the one-dimensional case.

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